Accelerated Life Models Under Progressive Type I Censoring

Murray D. Burke
Department of Mathematics & Statistics
University of Calgary
Calgary, Alberta, T2N 1N4
Canada
burke@math.ucalgary.ca

Abstract

Semiparametric additive accelerated life models are studied when failure times are subject to multistage progressive Type I censoring. At pre-determined times t_j, $j = 1, 2, \ldots, m$, R_j items are removed by the experimenter. The number R_j can be a pre-determined number or can be a pre-determined proportion of the items still functioning at t_j. In this paper, the martingale approach is taken to model the underlying counting processes. Large sample results are obtained for both progressively censored cases.

1 Introduction

Bagdonavičius and Nikulin (1999) have developed a unified approach to modeling failure time distributions when the items are subject to various stresses. In particular, if item i is subject to stress $\mathbf{z}$, a function of time t, the resulting failure time distribution function $S_{\mathbf{z}}$ is assumed to satisfy

$$H \circ S_{\mathbf{z}}(t) = H \circ S_0(t) + \int_0^t a(\mathbf{z}(s)) \; ds. \tag{1}$$

Here S_0 is an unknown baseline survival function. The function H is called the reliability generator; the inverse function H^{-1} is the resource. If $a(\mathbf{z}(s)) = \beta'\mathbf{z}(s)$, with β equal to a p-vector of unknown regression parameters, then one obtains a semiparametric additive accelerated life models as studied by Bordes (1999).

For example, if $H^{-1}(u) = \exp(-u^\gamma)$ is the Weibull resource, then $S_{\mathbf{z}}(t) = \exp\{-[(-\log S_0(t))^{1/\gamma} + \int \beta'\mathbf{z}(s)ds]^\gamma\}$, giving a cumulative hazard function $\Lambda_{\mathbf{z}}(t) = [(\Lambda_0(t))^{1/\gamma} + \int \beta'\mathbf{z}(s)ds]^\gamma$.

One of the aims in using such a model is to reduce the total time on test. In product testing, the cost of items to be tested may be high, the cost of running the test may be high, or the use of the testing facilities may be limited. Thus, by subjecting the test items to additional stresses, their lifetimes will be decreased. Inference about their original lifetime distribution can be obtained via (1). Another aim is to model the effects of different stresses that may occur in actual use.

In addition, accurate estimation of the left tail of the survival distribution is very important. To further reduce the total time on test, the experimenter can start with a large number of items. After a given period of time, say t_1, the experimenter would remove a number of functioning items from the test. Deliberate removal would be done at random and would only depend on the number of items still functioning at time t_1 and not on their subsequent survival times. This process of removal can occur at a finite number of times, t_j, $j = 1, \ldots, k$. For example, t_1 may be the warranty time of the product, that is, the maximum time when the customer can claim a full refund if the product fails. Time t_2 could be the maximum time for subsequent partial warranties, for example, those that cover parts but not labor. In addition, an experimenter may wish to remove a small number of items at other fixed times t_j in order to study the amount of wear. These removed items may be dissassembled and components may be subject to destructive testing.

The above (removal) censoring scheme is an instance of progressive multistage Type I censoring. Balakrishnan and Aggarwala studied progressive censoring schemes. They concentrated on Type II censoring where removal takes place at an order statistic of the failure time data. Their emphasis was on quantile estimation, estimation in parametric models and the order statistic properties of the data.

In reliability the experimenter can decide on the amount of stress $\mathbf{z}$ given to an item. The stresses can be considered nonrandom, but may be time dependent and are assumed to be left continuous. Suppose there are k different stresses, $\mathbf{z}_i(t)$, $1 \le i \le k$, in the experiment. The number of items subjected to stress $\mathbf{z}_i$ is equal to n_i and the failure time of the jth item in that group is labelled T_{ij}. Each item may be subject to random right censorship by L_{ij}. Such is the case when an item may accidentally be destroyed or missing. We assume the independence of $\{T_{ij}\}$ and $\{L_{ij}\}$.

In addition to the above random censoring by L_{ij}, at pre-determined times $t_{i\ell}$, where $t_{i1} < t_{i2} < \ldots < t_{im_i}$, $R_{i\ell}$ items are removed by the experimenter. We will consider two progressive censoring schemes:

Scheme 1: At time $t_{i\ell}$, randomly remove $[n_{i\ell}p_{i\ell}]$ of the items that are still functioning, where $n_{i\ell}$ is the number still functioning.

Scheme 2: At time $t_{i\ell}$, randomly remove $R_{i\ell}$ of the items that are still functioning.

Under both schemes, removal of items depends only on the set of indices of the items that are still functioning and not on their subsequent failure times. If item (i,j), (item j from group i) is removed at time $t_{i\ell}$ according to progressive scheme r, $r = 1,2$, put $C_{ij}^{(r)} = t_{i\ell}$. If item (i,j) is not progressively censored by scheme r at any of the times $t_{i\ell}$, we can put $C_{ij}^{(r)} = \tau_{S_i}$, where $\tau_{S_i} = \inf\{t : S_{z_i}(t) = 0\}$. We observe

$$X_{ij}^{(r)} = T_{ij} \wedge L_{ij} \wedge C_{ij}^{(r)}$$

as well as $I_{[X_{ij}^{(r)}=T_{ij}]}$, $I_{[X_{ij}^{(r)}=L_{ij}]}$, and $I_{[X_{ij}^{(r)}=C_{ij}^{(r)}]}$.

In this paper, the martingale approach is taken to model the underlying counting processes. Estimation of the unknown parameter β and of the baseline survival function S_0 is studied. Large sample results are obtained for both progressive censoring cases. In particular, it is shown that on a single probability space, the relevant martingales for the two progressive Type I schemes are asymptotically equivalent to the corresponding martingales under a scheme based on independent random right censoring. Thus, weak convergence results proved for independent random right censoring can be used to deduce those for progressively censored data. Some examples are given.

2 Estimation and Large Sample Theory

2.1 Martingale Results

Bordes (1999) considered the martingale

$$M_{ij}(t) = N_{ij}(t) - \int_0^t [\rho_0'(t)\psi(S_i(t))Y_{ij}(t) - \psi(S_i(t))Y_{ij}(t)\beta' z_i(t))]dt, \tag{2}$$

where $S_i = S_{z_i}$, $N_{ij}(t) = I_{[T_{ij} \le t,\ T_{ij} \le L_{ij}]}$; $Y_{ij}(t) = I_{[T_{ij} \ge t,\ L_{ij} \ge t]}$, $\rho_0(t) = H \circ S_0(t)$, and $\psi(u) = -(H^{-1})'(H(u))/u$.

For $r = 1,2$, let

$$Y_{ij}^{(r)}(t) = I_{[X_{ij}^{(r)} \ge t]}.$$

Then, the process incorporating both the random censoring above and progressive censoring scheme r can be represented as:

$$M_{ij}^{(r)}(t) = \int_0^t Y_{ij}^{(r)}(s) M_{ij}(ds), \tag{3}$$

which is (2) with Y_{ij} replaced by $Y_{ij}^{(r)}$.

Theorem 1. $M_{ij}^{(r)}$ is a martingale.

Consider the partial likelihood score function (see Bordes (1999), Bagdonavičius and Nikulin (1995), Lin and Ying (1994, 1995))

$$U^{(r)}(\beta,\tau) \quad = \quad \sum_{i=1}^{k} \int_0^{\tau} J^{(r)}(t)\{z_i(t) - \overline{z}(t, \widetilde{\mathbf{S}}^{(r)}(t))\} \widetilde{M}_i^{(r)}(\beta, dt);$$

$$\widetilde{M}_i^{(r)}(\beta, dt) \quad = \quad N_i^{(r)}(dt) - \beta' z_i(t) Y_i^{(r)}(t)\psi(\widetilde{S}_i^{(r)}(t))dt,$$

with

$$\overline{\mathbf{z}}(t, \mathbf{x}) = \frac{\sum_{i=1}^k \mathbf{z}_i(t) Y_i^{(r)}(t)\psi(x_i)}{\sum_{i=1}^k Y_i^{(r)}(t)\psi(x_i)}$$

and $\mathbf{x} = (x_1, \ldots, x_k)$, $\widetilde{\mathbf{S}}^{(r)} = (\widetilde{S}_i^{(r)}, \ldots, \widetilde{S}_i^{(r)})$ is the vector of product limit estimators of the survival function S_i.

Solving for β in $U^{(r)}(\beta, \tau) = 0$, one obtains

$$\begin{aligned} \widehat{\beta}^{(r)} \quad &= \quad \left(\sum_{i=1}^k \int_0^{\tau_i} J^{(r)}(t) \{\mathbf{z}_i(t) - \overline{\mathbf{z}}(t, \widetilde{\mathbf{S}}^{(r)}(t))\}^{\otimes 2} Y_i^{(r)}(t)\psi(\widetilde{S}_i^{(r)}(t))dt \right)^{-1} \\ &\times \sum_{i=1}^k \int_0^{\tau_i} J^{(r)}(t) \{\mathbf{z}_i(t) - \overline{\mathbf{z}}(t, \widetilde{\mathbf{S}}^{(r)}(t))\} N_i^{(r)}(dt), \end{aligned}$$

where $S_i(\tau_i) > 0$, $\mathbf{x}^{\otimes 2} = \mathbf{x}\mathbf{x}'$.

For scheme 2, let

$$p_{i\ell} = \frac{r_{i\ell}}{S(t_{i\ell})} \left(1 - \sum_{\ell'=1}^{\ell-1} \frac{r_{i\ell'}}{S(t_{i\ell'})} \right)^{-1},$$

For both schemes, we will assume

$$0 \le p_{i\ell} < 1, \text{ for } 1 \le i \le k, \quad 1 \le \ell \le m. \tag{4}$$

The following two theorems parallel those of Bordes (1999), who proved them under random right censoring.
Theorem 2. Then, under conditions (4) and A1-A5 of Bordes (1999), as n tends to infinity, $\sqrt{n}(\widehat{\beta}^{(r)} - \beta_0)$ is asymptotically normal with mean zero and covariance matrix Σ, $r = 1, 2$.
Theorem 3. Under the conditions above, as n tends to infinity,

$$\sqrt{n}(\widehat{S}_i^{(r)} - S_i) \rightarrow_D \xi \qquad in\ D[0, \tau_i],$$

where ξ is a mean-zero Gaussian process, where

$$\widehat{S}_i^{(r)}(t) = H^{-1} \left(\widehat{H \circ S}_0(t) + \int_0^t \widehat{\beta}^{(r)\prime} \mathbf{z}(u) du \right)$$

and $\widehat{H \circ S_0}$ is a baseline estimate of $H \circ S_0$, using the estimator $\widehat{\beta}^{(r)}$.

2.2 Progressive Censoring Mechanism

One way to represent these schemes is to consider, for each item (i, j), m independent uniform $(0, 1)$ random variables, $U_{ij\ell}$, $\ell = 1, 2, \ldots, m$, which are independent of the sequences $\{T_{ij}\}$ and $\{L_{ij}\}$. We will define the censoring random variables $C_{ij}^{(r)}$ for each scheme r, $r = 1, 2$.

Let $\rho_i^{(r)}(t_\ell, \omega) = \{j : t_\ell < T_{ij}(\omega),\ C_{ij}^{(r)}(\omega) \not< t_\ell\}$ denote the risk set of the items under censoring scheme r at time t_ℓ, $1 \le \ell \le m$. Let $n_{ri\ell} = \#\rho_i^{(r)}(t_\ell)$ denote the number of such items.

Scheme 1: For $r = 1$, item $(i, j) \in \rho_i^{(1)}(t_\ell)$ is removed, that is, $C_{ij}^{(1)} = t_\ell$, if $U_{ij\ell} \le U_{([n_{1i\ell} p_\ell])}$, where $U_{([n_{1i\ell} p_\ell])}$ is the $[n_{1i\ell} p_\ell]$-order statistic from the set $\{U_{ij\ell} : i \in \rho_i^{(1)}(t_\ell, \omega)\}$. Put $C_{ij}^{(1)} = \tau_{S_i}$, if item (i, j) is not censored under scheme 1.

Scheme 2: For $r = 2$, item $(i, j) \in \rho_i^{(2)}(t_\ell, \omega)$ is removed, that is, $C_{ij}^{(2)} = t_\ell$, if $U_{ij\ell} \le U_{(R_\ell)}$, where $U_{(R_\ell)}$ is the R_ℓth-order statistic from the set $\{U_{ij\ell} : i \in \rho_i^{(2)}(t_\ell, \omega)\}$. Put $C_{ij}^{(2)} = \tau_{S_i}$, if item (i, j) is not censored under scheme 2.

Then, the resulting $N_{ij}^{(r)}(t) = I[T_{ij} \le t,\ T_{ij} \le C_{ij}^{(r)} \wedge L_{ij}]$, $Y_{ij}^{(r)}(t) = I[T_{ij} \wedge C_{ij}^{(r)} \wedge L_{ij} \ge t]$ and $M_{ij}^{(r)}$, defined by (3), $j = 1, 2, \ldots, n_i$, $i = 1, 2, \ldots k$, have the required joint distributions for scheme r, $r = 1, 2$.

We introduce a third scheme, which is a random censoring scheme, based on independent random variables $C_{ij}^{(3)}$, having discrete probability mass function

$$
\begin{aligned}
P[C_{ij}^{(3)} &= t_1] = p_{i1} \\
P[C_{ij}^{(3)} &= t_\ell] = p_{i\ell} \prod_{\ell'=1}^{\ell-1} (1 - p_{i\ell'}), \quad \ell = 2, \ldots, m \\
P[C_{ij}^{(3)} &= \tau_{S_i}] = 1 - \sum_{\ell=1}^{m} P[C_{ij}^{(3)} = t_\ell]
\end{aligned}
$$

and define $N_{ij}^{(3)}(t) = I[T_{ij} \le t,\ T_{ij} \le C_{ij}^{(3)} \wedge L_{ij}]$, $Y_{ij}^{(3)}(t) = I[T_{ij} \wedge C_{ij}^{(3)} \wedge L_{ij} \ge t]$, $j = 1, 2, \ldots, n_i$, $i = 1, 2, \ldots k$, as above ($r = 3$). In terms of the above construction,

Scheme 3: For $r = 3$, item $(i, j) \in \rho_i^{(3)}(t_\ell, \omega)$ is removed, that is, $C_{ij}^{(3)} = t_\ell$, if $U_{ij\ell} \le p_{i\ell}$. Otherwise, put $C_{ij}^{(3)} = \tau_{S_i}$.

Theorems 1-3 will follow from: Under the conditions above,

$$
\sup_{0 \le t \le \tau_{S_i}} \left| \frac{\Sigma_{j=1}^{n_i} Y_{ij}^{(r)}(t)}{n} - \frac{\Sigma_{j=1}^{n_i} Y_{ij}^{(3)}(t)}{n} \right| =_{a.s.} o(n^{-1/2+\varepsilon}), \quad r = 1, 2
$$

and

$$
n^{-1/2} \sup_{0 \le t \le \tau_{S_i}} \left| D_n^{(r)}(t) \right| \to_P 0,
$$

where

$$
D_n^{(r)}(t) = \sum_{j=1}^{n_i} (M_{ij}^{(r)}(t) - M_{ij}^{(3)}(t)), \quad r = 1, 2.
$$

Also, from Burke (2004), for the product limit estimators,

$$
\sup_{t \le \tau_i} \sqrt{n_i} \left| \widetilde{S}_i^{(r)}(t) - \widetilde{S}_i^{(3)}(t) \right| \to_P 0.
$$

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